MTH 301: Group Theory

Practice Assignment III

1. Show a finite group G cannot be simple, if:
   (a) $|G| = p^k$, where p is a prime and $k > 1$.
   (b) $|G| = 2p^k$, where p is a prime and $k \geq 1$.
   (c) $|G| = pq$, where the p and q are distinct primes.

2. Let $|G| = pq$, where p and q are distinct primes. Then show that

   $$G \cong H \ltimes K,$$

   where $H \in \mathrm{Syl}_p(G)$ and $K \in \mathrm{Syl}_q(G)$. [Hint: Use Problem 4 from Practice Assignment III.]

3. Show that, up to isomorphism, there is a unique non-abelian group of order 10. nd ord

4. Establish the non-simplicity of all groups of non-prime order less than 60.

5. Show that if $|G| = 60$ and $n_5 > 1$, then G is simple. [Hint: Assume on the contrary that G has a proper normal subgroup H. Then $n_5 = 6$ and for any Sylow 5-subgroup P, $|N_G(P)| = 10$. Now you will need to use the non-simplicity of groups of order 15, 20 and 30, and argue.]

6. Show that A_5 is simple. [Hint: Show that A_5 has at least two distinct Sylow 5-subgroups.]

7. Compute $\mathrm{Aut}(G)$ for the following groups.
   (a) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (b) $G = D_8$.
   (c) $G = Q_8$.

8. Prove Q_8 is an extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$.

9. Does there exist group of order 12 isomorphic to a non-trivial semi-direct product of the form $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_3$ or $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

10. If P is a Sylow p-subgroup and Q a p-subgroup of a finite group G, then show that

    $$N_G(P) \cap Q = P \cap Q.$$

    [Hint: It suffices to show that $N_G(P) \cap Q \leq P$.]